A RENEWAL FUNCTION ARISING IN WARRANTY ANALYSIS[1]

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1. INTRODUCTION

Since a real or potential cost is involved, any item sold with a warranty must necessarily be priced higher than if it were sold without a warranty. How much more the seller should charge and how much more the buyer should be willing to pay depends upon the structure of the warranty and the life distribution of the item. An analysis of pro rata and free-replacement warranties from both buyer's and seller's points of view is given by Blischke and Scheuer (1975 and 1977).

In this paper we shall consider only the free-replacement warranty and shall be concerned mainly with the seller's (or supplier's, manufacturer's, and so forth) point of view. Of primary importance from this point of view is the long-run profitability of the item.

An important consideration in analyzing long-run profits for items sold under a free-replacement warranty is the expected income over the life cycle of the item. This, of course, is a function of the expected number of replacement items sold over the life cycle. (We assume here that the buyer purchases an identical replacement when the item in service at the end of the warranty period fails and, for simplicity, that the purchase and initiation of operation of the replacement are instantaneous.) This expected number, called the "renewal function" for the associated random variable, is the subject of this investigation.

In the ensuing, we shall discuss in more detail the nature of the free-replacement warranty and its associated costs/profits, the role of renewal theory in analyzing warranty policies, and the specific renewal function encountered in the context just described.

The form of a renewal function depends ultimately on the underlying life distribution of the items in question. Typically in dealing with renewal functions, closed form expressions are available only for a few special cases, although limiting results are quite generally available. We shall find this to be true of the special renewal function under consideration here as well. Analytical results will be given for the exponential distribution and (although admittedly of limited interest as a distribution of lifetimes) for the uniform distribution. Some preliminary results of a numerical investigation of the special renewal function for gamma and Weibull distributed lifetimes will also be discussed.

2. THE SPECIAL RENEWAL FUNCTION AND ITS ROLE IN THE ANALYSIS OF WARRANTY POLICIES

2.1 The Analysis of Warranty Policies. In the analysis of warranty policies given by Blischke and Scheuer (1975 and 1977) the basic considerations were the comparison of cost to the consumer,

and of profit to the supplier, of warrantied versus unwarrantied items. In the present paper we shall limit consideration to the point of view of the supplier. From his point of view, the cost comparison leads to the establishment of a differential pricing structure which will equate expected long-run profit in the two situations. Profit, of course, is a function of cost and income. In our previous work (Blischke and Scheuer (1975)) we derived the expected profit per warranty cycle. Here we are concerned with the long-run profit over the life cycle of the item. This can be approximated for relatively long life cycles by pursuing an analysis along the lines of our 1975 paper. (See especially Sections 2.1.1 and 2.2). Our present objective is to obtain an exact expression for this quantity. A result of this type will also provide a basis for evaluation of the adequacy of the approximation.

2.1.1 The Free-Replacement Warranty. The specific warranty policy under consideration here is the free-replacement policy. Under a warranty of this type the supplier provides replacements for failed items free of charge until a specified period of service, W, is attained. His income during this period is the price, C, charged for the initial item. His expected cost is the sum of the cost of supplying the initial item and the expected cost of all replacements required to provide the total warrantied service time, W. In the sequel we shall express this expected cost, following Blischke and Scheuer (1975), as $g[1 + M_X(W)]$, where g is the cost per unit, X is the random lifetime of an individual item and $M_X(W)$ is the associated renewal function evaluated at W. (In this expression the quantity $1 + M_X(W)$ is the expected total number of items supplied, that is, the initial item plus the expected number of replacements.)

2.1.2 The Excess Random Variable. For the long-run analysis of the free-replacement warranty policy, it is important to note that no cost is incurred and no income obtained after W until the item in service at time W fails. The symbol Y is used to denote the random time at which this event takes place. This can also be expressed as $Y = W + \gamma(W)$, where $\gamma(W)$, the "excess random variable," is the (random) residual lifetime of the item in service at time W. This random variable is key to the analysis which follows. It is also called the "excess life" or "residual life" (Ross (1970), p.44), "remaining life" (Barlow and Proschan (1975), p. 168) and "forward recurrence time" or "residual life-time" (Cox (1962), p. 27) and has some unusual properties. (See, for example, Feller (1966), Sections I.4 and VI.7.)

2.2 The Role of Renewal Functions. In the foregoing we have seen that the renewal function, $M_X(\cdot)$, of the basic lifetime random variable, X, plays an important role in determining expected
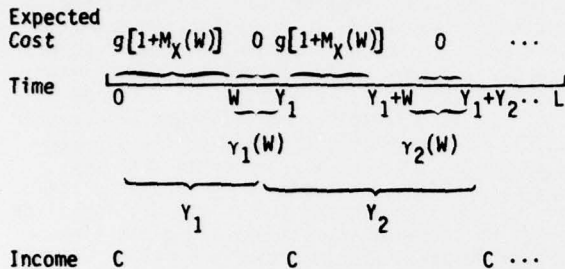
profit on a per-cycle basis. In particular, expected profit per cycle is $P = C - g[1 + M_X(W)]$.

We turn now to the analysis of long-run expected profit. In this case we look at repetitions of the warranty cycles. The first such cycle extends from 0 to $Y_1 = W + \gamma_1(W)$, say, the second from Y_1 to Y_2, and so forth. Schematically, we have

Expected
Cost $\quad g[1 + M_X(W)] \quad 0 \quad g[1 + M_X(W)] \quad 0 \quad \cdots$

Time $\quad \overline{\underset{0}{\rule{0pt}{0pt}} \quad \underset{W \ \ Y_1}{\rule{0pt}{0pt}} \quad \underset{Y_1 + W \ \ Y_1 + Y_2}{\rule{0pt}{0pt}} \cdots L}$

$\qquad\qquad \gamma_1(W) \qquad\qquad \gamma_2(W)$

$\qquad\qquad\quad Y_1 \qquad\qquad\qquad Y_2$

Income $\quad C \qquad\qquad C \qquad\qquad C \cdots$

The total expected profit is thus seen to be P times the number of expected repetitions of this process over the life cycle, L. This quantity is precisely the renewal function of the random variable Y, evaluated at L. We call this the special renewal function and denote it $M_Y(\cdot)$. We note, incidentally, that in our previous work we approximated this by the quantity $L/E(Y)$. In some examples this approximation was found to be inadequate, which motivated the present study.

The remainder of this paper is devoted to an evaluation of $M_Y(\cdot)$. This function and $M_Y(\cdot)$ are all that are required to complete the cost analysis.

3. ANALYTICAL INVESTIGATION OF $M_Y(\cdot)$.

3.1 General Renewal Theoretic Results.
We begin with the basic renewal process involving a single warranty cycle. X, Y, $\gamma(\cdot)$, W and L are as defined previously. Let $X_1, X_2, \ldots$ be the lifetimes of the individual items within a warranty cycle. We assume that $X_1, X_2, \ldots$ are nonnegative random variables which are independent and identically distributed with CDF $F_X(\cdot)$. We write $S_n = \sum_{i=1}^{n} X_i$ ($i = 1, 2, \ldots$), $S_0 = 0$, and $\mu = E(X)$. For any CDF, $F(\cdot)$, we define $F^{(n)}(\cdot)$ to be the n-fold convolution of $F(\cdot)$ with itself, with

$$F^{(0)}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0, \end{cases}$$

$N = N(t)$ = number of replacements required in the interval $[0,t]$, and $M(t) = E(N(t))$.

A well-known general renewal-theoretic result is that
(3.1) $\quad P(N(t) = n) = F^{(n)}(t) - F^{(n+1)}(t)$.
This provides an immediate expression for $M_X(t)$ in terms of the convolutions $F_X^{(n)}(t)$ and for $M_Y(t)$ in terms of $F_Y^{(n)}(t)$. We turn next to the problem of determining $F_Y(\cdot)$ and $F_Y^{(n)}(\cdot)$.

Many asymptotic results regarding renewal functions are also available. Of primary interest here is the Elementary Renewal Theorem (Ross (1970), p. 40), which was used in our previous

work to approximate $M_Y(L)$. By this theorem, $M_Y(t)/t \to 1/E(Y)$ as $t \to \infty$. (It can also be shown (cf. Smith (1958), equation (1.1)) that $E(Y) = \mu[1 + M_X(W)]$, which was the result actually used in our approximation.)

3.2 Distribution of Y.

3.2.1 Distribution of the Excess Random Variable.
Since $Y = W + \gamma(W)$, the distribution of Y is simply a translation of the distribution of the excess random variable. Thus the fundamental result required is the distribution of $\gamma(W)$. There are several ways of expressing this result. All, of course, relate back to the basic distribution of X since we can also write $\gamma(W)$ as $\gamma(W) = S_{N_X(W)+1} - W$.

The distribution of γ is shown by Ross (1970, Section 3.6) to be
(3.2) $\quad F_{\gamma(W)}(t) = F_X(W+t) - \int_0^W [1 - F_X(W+t-u)] \, dM_X(u)$.
An equivalent expression in terms of the corresponding densities f and m (when they exist) is given by Cox (1962, Section 5.2) as
(3.3) $\quad f_{\gamma(W)}(t) = f_X(W+t) + \int_0^W m(W-u) f_X(u+t) du$.

3.2.2 Mixture Representation.
It is of interest to note that in addition to these classical representations, the distribution of the excess random variable can also be expressed as a mixture of distributions (cf. Blischke (1965) and (1968)), namely
(3.4) $\quad F_{\gamma(W)}(t) = \sum_{n=0}^{\infty} P\{\gamma(W) \leq t | N(W) = n\} P\{N(W) = n\}$.
Here the distribution of N (given in equation (3.1)) is the mixing distribution and the conditional distributions of γ given N are the components of the mixture. The difficulty here is the evaluation of these conditional distributions. Since the event $\{N(W) = n\}$ is equivalent to the event $\{S_n < W, S_{n+1} \geq W\}$, the conditional distributions become
$\quad P\{\gamma(W) \leq t | N(W) = n\} = P\{S_{n+1} \leq W+t | S_n < W, S_{n+1} \geq W\}$
which can be expressed as an integral over the appropriate region of the bivariate distribution of S_n, S_{n+1}. Except for a few simple cases, these integrals are tedious to evaluate.

One property often encountered in dealing with mixed distributions is that they may be multi-modal. This is indeed the case for the distribution of the excess random variable, a fact that became quite apparent in some of our computer simulations.

Another property of mixtures of the type we are dealing with here is that the moments of the mixed distribution can be expressed as weighted averages of the moments of the components. We have not pursued this point but it would be of interest in some applications. (For example, on an individual basis one would be interested in the conditional expected lifetime of the item actually in service at the end of the warranty period.)

3.3 Examples.

3.3.1 The Exponential Distribution.

For the exponential distribution,

$$(3.5) \qquad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases},$$

explicit expressions for all of the above are easily obtained. We use (3.3) to obtain the density of the excess random variable. The "renewal density" is $m(t) = 1/E(X) = \lambda$. (See Cox (1960), p. 30.) Thus, for $t>0$,

$$f_{\gamma(W)}(t) = \lambda e^{-\lambda(W+t)} + \int_0^W \lambda^2 e^{-\lambda(u+t)} du = \lambda e^{-\lambda t},$$

which is, of course, a well-known result. The density of Y is simply a translated exponential. The n-fold convolution of this is a translated gamma, with CDF

$$(3.6) \quad F_Y^{(n)}(y) = \begin{cases} 0 & y<nW \\ 1 - \sum_{i=0}^{n-1} \dfrac{[\lambda(y-nW)]^i}{i!} e^{-\lambda(y-nW)} & y \geq nW. \end{cases}$$

In writing the renewal function, it will be convenient to express L as a multiple of W, say $L = \ell W$. We then obtain, from equations (3.1) and (3.6),

$$P(N_\gamma(\ell W)=n) = e^{-\lambda(\ell-n-1)W} \sum_{i=0}^n \frac{\lambda^i (\ell-n-1)^i W^i}{i!}$$
$$- e^{-\lambda(\ell-n)W} \sum_{i=0}^{n-1} \frac{\lambda^i (\ell-n)^i W^i}{i!}$$

for $n = 0,1,\ldots,\ell-1$. Finally, the special renewal function is found to be

$$(3.7) \quad M_\gamma(\ell W) = (\ell-1) - \sum_{j=1}^{\ell-1} e^{-j\lambda W} \sum_{i=0}^{\ell-1-j} \frac{(j\lambda W)^i}{i!} .$$

3.3.2 The Uniform Distribution.

Although the uniform distribution is admittedly of limited interest as a life distribution, it is a convenient and nontrivial example to illustrate the mixture formulation. (In the exponential case, all components of the mixture are the initial distribution of (3.5).) Take

$$(3.8) \qquad f_X(x) = \begin{cases} \dfrac{1}{\theta} & 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

It seems sensible to assume that $\theta > W$ since otherwise replacements are required with probability one.

We proceed to express the distribution of the variable as a mixture. The mixing distribution is the distribuiton of N. We find

$$P(N=0) = P(X_1>W) = (\theta-W)/\theta ,$$

$$P(N=1) = P(X_1<W,\ X_1+X_2>W)$$
$$= \int_0^W \int_{W-x_1}^\theta \left(\frac{1}{\theta^2}\right) dx_1 dx_2 = \frac{2\theta W - W^2}{2\theta^2} ,$$

and so forth. The general result is

$$(3.9) \quad P(N=n) = \frac{(n+1)\theta W^n - W^{n+1}}{(n+1)!\theta^{n+1}} = \frac{W^n}{n!\theta^{n+1}} \left(\theta - \frac{W}{n+1}\right).$$

The components of the mixture are the conditional distributions of $\gamma(W)$ given $N=n$. These are

$$(3.10) \quad f_\gamma(t|N=n) = \begin{cases} \dfrac{1}{\theta-\frac{W}{n+1}} & 0 < t < \theta-W \\[2ex] \dfrac{W^n-(W+t-\theta)^n}{\theta W^n - \frac{W^{n+1}}{n+1}} & \theta-W < t < \theta. \end{cases}$$

From equations (3.9) and (3.10), we find the unconditional distribution of the excess random variable to be

$$(3.11) \quad f_{\gamma(W)}(t) = \begin{cases} \displaystyle\sum_{n=0}^\infty \frac{1}{\theta-\frac{W}{n+1}} \frac{W^n}{n!\theta^{n+1}}\left(\theta-\frac{W}{n+1}\right) & 0<t<\theta-W \\[2ex] \displaystyle\sum_{n=0}^\infty \frac{W^n-(W+t-\theta)^n}{\theta W^n - \frac{W^{n+1}}{n+1}} \frac{W^n}{n!\theta^{n+1}}\left(\theta-\frac{W}{n+1}\right) & \\ & \theta-W<t<\theta \end{cases}$$

$$= \begin{cases} \dfrac{1}{\theta} e^{W/\theta} & 0<t<\theta-W \\[2ex] \dfrac{1}{\theta} e^{W/\theta} - \dfrac{1}{\theta} e^{(W+t-\theta)/\theta} & \theta-W<t<\theta, \end{cases}$$

from which it follows that the distribution of Y is

$$f_Y(y) = \begin{cases} \dfrac{1}{\theta} e^{W/\theta} & W<y<\theta \\[2ex] \dfrac{1}{\theta} e^{W/\theta} - \dfrac{1}{\theta} e^{(y-\theta)/\theta} & \theta<y<\theta+W. \end{cases}$$

The conditional means of the excess random variable are easily seen to be

$$E(\gamma(W)|N=n) = \frac{\theta}{2} - \frac{\frac{W}{n+1}\left(\theta - \frac{2W}{n+2}\right)}{2\left(\theta - \frac{W}{n+1}\right)} .$$

One can use this result or equation (3.11) to obtain the unconditional mean of $\gamma(W)$. The result is $E(\gamma(W)) = (\theta/2)e^{W/\theta} - W$, from which it follows immediately that $E(Y) = (\theta/2)e^{W/\theta}$.

The convolutions of f_Y are rather tedious and we have not pursued the exact analysis. One could, however, use the above result and the Elementary Renewal Theorem to obtain an asymptotic expression for $M_\gamma(\cdot)$.

3.3.3 The Gamma and Weibull Distributions.

The gamma and Weibull distributions, with respective densities

$$(3.11) \quad f_X(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-x/\beta} & x \geq 0 \\ 0 & x<0 \end{cases}$$

and

$$(3.12) \quad f_X(x) = \begin{cases} \dfrac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-(x/\beta)^\alpha} & x \geq 0 \\ 0 & x<0, \end{cases}$$

are two of the more interesting life distributions. Unfortunately both are often difficult to deal with analytically. This is certainly the case here; general, closed-form expressions for the basic renewal functions, $M_Y(\cdot)$, to say nothing of the special renewal functions, $M_Y(\cdot)$, exist for neither. (There is however, a closed-form expression for the basic renewal function for the gamma distribution if the shape parameter, α, is integer-valued. See, for example, Barlow and Proschan (1965), page 57.) It follows that either asymptotic results or numerical approximations are required. The basic renewal function has been evaluated numerically and tabulated to some extent by Soland (1968). This may be used to approximate $M_Y(L)$ for relatively large L as indicated previously.

4. NUMERICAL INVESTIGATION

4.1 Structure of the Numerical Studies.

Because of the complexity encountered in the analytical investigation of the distribution of the excess random variable and the evaluation of the special renewal function, programs were written to provide an opportunity to investigate the properties of both of these numerically. The basic life distributions that can be used in the simulations with these programs are the exponential, gamma, Weibull, uniform and normal. Here we shall concern ourselves only with the gamma and Weibull distributions. Some preliminary results concerning the special renewal function for these will be discussed below. The purposes of the special renewal program were to provide a means of investigating the approximation $M_Y(L)/L \sim 1/E(Y)$, and to provide a means of evaluating $M_Y(L)$ when the approximation is not adequate.

As noted previously, $E(Y) = \mu\left[1 + M_Y(W)\right]$. $M_X(\cdot)$ is tabulated by Soland (1968) for the gamma and Weibull distributions for several choices of α and β selected so that in all cases $\mu=1$. We restrict consideration to cases of this type as well. The specific results which will be reported are for $\alpha=2,3,4,5$ in combination the appropriate values of β so that $\mu=1$ in each case. All combinations of W = 0.5, 1.0 and 1.5 with L=5, 10 and 15 were used. (This gave warranty periods less than, equal to, and greater than the mean life and life cycles ranging from 3+ to 30 times the warranty period.) In each simulation 500 repetitions of the special renewal process were performed.

4.2 Results.

In each of the simulations the average number of renewals, say $\hat{M}_Y(L)$ was calculated (along with certain additional relevant summary statistics). The basic results for the gamma distribution are given in Table 1 and for the Weibull distribution in Table 2. In each case the values tabulated are $\hat{M}_Y(L)/L$. For comparison purposes, values of $1/E(Y)$, based on Soland's tables of $M_X(W)$, are included as well.

Table 1
Values of $\hat{M}_Y(L)/L$ for the Gamma Distribution

Parameters				W = 0.5		
α	β	1/E(Y)	L	5	10	15
2	.50	.779		.707	.736	.753
3	.33	.836		.750	.794	.805
4	.25	.874		.792	.830	.840
5	.20	.902		.819	.858	.868
				W = 1.0		
2	.50	.570		.491	.526	.539
3	.33	.601		.522	.561	.573
4	.25	.618		.533	.578	.588
5	.20	.628		.539	.587	.602
				W = 1.5		
2	.50	.444		.350	.394	.413
3	.33	.462		.369	.410	.428
4	.25	.471		.377	.424	.440
5	.20	.476		.380	.428	.443

Table 2
Values of $\hat{M}_Y(L)/L$ for the Weibull Distribution

Parameters				W = 0.5		
α	β	1/E(Y)	L	5	10	15
2	1.13	.844		.757	.799	.814
3	1.12	.920		.832	.875	.888
4	1.10	.960		.869	.912	.927
5	1.09	.980		.882	.932	.947
				W = 1.0		
2	1.13	.616		.532	.574	.593
3	1.12	.653		.568	.612	.625
4	1.10	.668		.575	.622	.641
5	1.09	.675		.581	.633	.646
				W = 1.5		
2	1.13	.469		.372	.421	.436
3	1.12	.480		.388	.431	.446
4	1.10	.482		.396	.431	.449
5	1.09	.485		.397	.432	.453

In the simulations we also calculated the sample variances of the number of renewals. From these results one can estimate the standard error of $\hat{M}_Y(L)/L$. The results ranged from less than .002 to .009, with all standard errors except those for combinations of the smallest values of W and L less than .005. Given that the accuracy of the computer simulations themselves is adequate, one can therefore conclude that we have the second digit determined to within one unit or so except for a few cases.

4.3 Discussion.

It is important to note that the approximation based on E(Y) is quite inaccurate: $1/E(Y)$ always overestimates $\hat{M}_Y(L)$, with the difference, of course, decreasing as L increases. (Thus $L/E(Y)$ would consistently overestimate $\hat{M}_Y(L)$

which would lead to an overestimate of the expected income over the life cycle of the item.) We note, however, that there is some consistency in the amount of overestimation. The average amounts by which $1/E(Y)$ exceeds $\hat{M}_Y(L)/L$ for the Weibull distribution with L = 5, 10, and 15 are .031, .047, and .090, respectively. For the gamma distribution the corresponding values are .031, .044, and .086. In each case these values are slightly less than .5/L. (Overall, about .45/L). This suggests that $1/E(Y)$ estimates approximately $[.5 + M_Y(L)]/L$. In other runs this pattern prevailed; in fact, the difference appears to approach .5/L more closely as L increases. For example, for the gamma distribution with $\alpha=3$ and $\beta=1/3$, $1/E(Y)$ is .4616. In a run of 1,000 repetitions with W=1.5 and L=99 we obtained $\hat{M}_Y(99)$ = 45.213, giving $[.5 + M_T(99)]/99$ = .4617!

There remain a number of unanswered questions: What is the appropriate correction for smaller L? How is it related to α, β, and W? How about non-integer values of α? Values of μ other than 1? and so forth. Our tentative conclusion, however, is that a better approximation to $M_Y(L)$ is $L/E(Y) - .5$. This should be quite accurate for large L and will be a conservative estimate (which is usually desirable when predicting income) for smaller values of L.

We note, finally, that there is some logic to this result. The quantity $1/E(Y)$ should be the long-run average number of warranty cycles per unit of time. This appears to approximate $[.5 + M_Y(L)]/L$. For these to agree, the numerator of the last expression should be the expected number of warranty cycles in L. This can be interpreted in that way if we note that the total expected number of sales is $M_Y(L) + 1$ (where, again, the "1" is the initial sale and M is the number of renewals), but adjust this downward by .5 because in the overall aggregate we expect to be right at the middle of a warranty cycle at time L.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A RENEWAL FUNCTION ARISING IN WARRANTY ANALYSIS.		5. TYPE OF REPORT & PERIOD COVERED Technical reptis
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Wallace R. Blischke Ernest M. Scheuer		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0733
9. PERFORMING ORGANIZATION NAME AND ADDRESS Decision Sciences Department University of Southern California Los Angeles, CA 90007		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR042-323
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics and Probability Program, Code 436 Arlington, Virginia 2217		11. REPORT DATE 1977
		13. NUMBER OF PAGES 5
14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office) 7 p.		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Renewal functions   Gamma and Weibull
Warranty Cost   distributions
Life-Cycle Costs   Uniform distribution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Under a free-replacement warranty of duration W, the customer is provided, for an initial cost of C, as many replacement items as needed to provide service for a period W. Payments of C are not made at fixed intervals of length W, but in random cycles of length $Y = W + \gamma(W)$, where $\gamma(W)$ is the (random) remaining life-time of the item in service W time units after the beginning of a cycle. The expected number of payments over the life cycle, L, of the item is given by $M_Y(L)$, the renewal function for the random variable Y. We investigate this

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renewal function analytically and numerically and compare our findings with known asymptotic results. The distribution of Y, and hence the renewal function, depends on the underlying failure distribution of the items. Several choices for this distribution, including the gamma and Weibull, are considered. This continues work begun in : Blischke, Wallace R., and Ernest M. Scheuer, "Calculation of the Cost of Warranty Policies as a Function of Estimated Life Distributions," Naval Research Logistics Quarterly, 22, 681-696 (1975).